Differential Equations

Jeevitha N
Department of Mathematics

Introduction

Differential equation : An equation which involves independent variable, dependent variable and derivative of dependent variable with respect to independent variable is known as differential equation.

Ex : $\dfrac{dy}{dx} + x^2 y = x$, $\quad y\dfrac{dy}{dx} = x$

Order of a differential equation : Order of a differential equation is defined as the order of the highest derivative involved in the differential equation.

Ex : 1. $\dfrac{dy}{dx} + x^2 y = x$, $\quad order = 1$

2. $\dfrac{d^2y}{dx^2} + x^2 \dfrac{dy}{dx} = 0$, $\quad order = 2$

Degree of a differential equation : Degree of a differential equation is defined as the integral power of the highest order derivative involved in the equation.

Ex :

1. $$x^2\frac{dy}{dx} + xy = y\,, \quad degree = 1$$

2. $$\left(\frac{d^3y}{dx^3}\right)^2 + y\left(\frac{dy}{dx}\right)^3 = x^2, \quad degree = 2$$

order

$$\left(\frac{d^3y}{dx^3}\right)^2 + y\left(\frac{dy}{dx}\right)^3 = x^2,$$

degree

Methods of solving first order and first degree differential equations

- Variable separable
- Homogeneous equation
- Linear differential equation
- Equations reducible to linear form
- Exact equations

Variable separable:

A first order differential equation $\frac{dy}{dx} = f(x, y)$ is called a separable equation if the function f(x , y) can be factored into the product of two functions of x and y i.e., $f(x, y) = p(x)h(y)$
where p(x) and h(y) are continuous functions.
Considering

$$\frac{dy}{dx} = p(x)h(y) \Rightarrow \frac{dy}{h(y)} = p(x)dx \quad \text{(separating the variables)}$$

On integration,

$$\int \frac{dy}{h(y)} = \int p(x)dx + c \quad \text{where 'c' is integration constant}$$

$$H(y) = P(x) + c$$

This represents the general solution of the differential equation

Ex: 1. solve $y\frac{dy}{dx} - x = 0$ by separating variables

Soln:

$$y\frac{dy}{dx} - x = 0$$

$$\Longrightarrow \quad y\frac{dy}{dx} = x$$

$$\Longrightarrow \quad ydy = xdx$$

On integrating,

$$\int ydy = \int xdx + \frac{c}{2}$$ integration constant

$$\Longrightarrow \quad \frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2}$$

$\Longrightarrow \quad y^2 = x^2 + c$ is the required solution.

Homogeneous Differential Equation

A first order differential equation $\frac{dy}{dx} = f(x, y)$ is called homogeneous equation, if f(x , y) is a homogeneous function of the zero order.

A homogeneous equation can be solved by substituting y = ux(where u is a new function depending on x) which leads to a separable differential equation. On solving using separable method the general solution is obtained.

Ex: Solve the differential equation $(2x + y)dx - xdy = 0$

Soln: $$\frac{dy}{dx} = \frac{2x + y}{x}$$

Here $f(x, y) = \frac{2x + y}{x}$ is a homogeneous function of zero order

put y = ux,

$$\Longrightarrow dy = d(ux) = udx + xdu$$

Substituting this into the differential equation, we obtain

$$\Longrightarrow (2x + ux)dx - x(udx + xdu) = 0$$

$$\Longrightarrow 2xdx + u\cancel{x}dx - u\cancel{x}dx - x^2\, du = 0$$

Dividing both sides by x gives :

$$\Longrightarrow xdu = 2dx$$

Separating the variables $\Longrightarrow du = \frac{2}{x}dx + c$

On integrating , $\int du = \int \frac{2}{x}dx + c$

$$\Longrightarrow u = 2logx + c$$

Substituting $u = \frac{y}{x}$ we get $y = 2xlogx + cx$

Which is the required solution

Linear Differential Equation

A differential equation of type $\frac{dy}{dx} + P(x)y = Q(x)$ where P(x) and Q(x) are functions of x or constants, is called a linear differential equation of first order.

using Integrating factor(IF) method, we obtain the solution i.e., IF = $e^{\int pdx}$ and solution is given by

$$y(IF) = \int Q(x)(IF)dx + c$$

Ex : solve $\frac{dy}{dx} + \frac{2}{x}y = x^3$

Soln : The given equation is of the form $\frac{dy}{dx} + P(x)y = Q(x)$

Here $P(x) = \frac{2}{x}$ $and\ Q(x) = x^3$

Integrating factor IF = $e^{\int p dx} = e^{\int \frac{2}{x} dx} = e^{2logx} = e^{logx^2} = x^2$

Hence the general solution is

$$y(x^2) = \int x^3 x^2 dx + c$$

$$\Longrightarrow yx^2 = \int x^5 dx + c$$

$$\Longrightarrow yx^2 = \frac{x^6}{6} + c \;\; or \;\; y = \frac{x^4}{6} + \frac{c}{x^2}$$ is the required solution

Bernoulli's Equation(Reducible to linear)

An equation of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ ... ... (1) (n $\neq 0, n \neq 1$) where P and Q are Constants or functions of x alone is called Bernoulli Equation.

Dividing (1) throughout by y^n, we get $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$

Put $y^{1-n} = v$ then $(1 - \mathrm{n})\hat{}\mathrm{y}^{-\mathrm{n}}\frac{dy}{dx} = \frac{dv}{dx}$ and equation (2) becomes

$$\frac{1}{1-n}\frac{dv}{dx} + Pv = Q \text{ or } \frac{dv}{dx} + P(1-n)v = Q(1-n) \ldots\ldots (3)$$

This a linear equation with v as the dependent variable and may be solved by finding integrating factor.

Ex : 1. Solve $\dfrac{dy}{dx} - \dfrac{y}{x} = y^2$

Soln : The given equation is Bernoulli's equation. Dividing throughout by y^2

$$\Longrightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{y^{-1}}{x} = 1 \; or \; y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = 1$$

Put

$$y^{-1} = v \; so \; that - y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$$

so

$$-\frac{dv}{dx} - \frac{1}{x}v = 1 \quad or \quad \frac{dv}{dx} + \frac{1}{x}v = -1 \qquad \text{(linear equation)}$$

Here $\quad P = \dfrac{1}{x} \qquad Q = -1$

Integrating factor, $e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{log x} = x$

solution is given by $vx = \displaystyle\int (-1)(x) dx + c$

$$vx = -\frac{x^2}{2} + c \quad or \;\; y^{-1}x = -\frac{x^2}{2} + c$$

2. Solve $\frac{dy}{dx} - 2ytanx = y^2 \tan^2 x$

Soln : Dividing throughout by y², $y^{-2}\frac{dy}{dx} - 2y^{-1}tanx = \tan^2 x$

put $y^{-1} = v \;\; so\, that\, -y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

$$\therefore -\frac{dv}{dx} - 2vtanx = \tan^2 x$$

$$\frac{dv}{dx} + 2vtanx = -\tan^2 x$$

where $P = 2tanx \;\; Q = -\tan^2 x$

$$IF\ e^{\int pdx} = e^{\int 2tanxdx} = e^{2logsecx} = e^{logsec^2x} = \sec^2 x$$

$$\therefore vsec^2x = -\int \tan^2 x \sec^2 x\, dx + c$$

$$\therefore vsec^2x = -\frac{\tan^3 x}{3} + c$$

$\therefore \frac{1}{y}sec^2x = -\frac{\tan^3 x}{3} + c$ is the solution

Exact Differential Equation

A differential equation of the type $M(x,y)dx + N(x,y)dy = 0 \dots \dots (1)$ is said to be an exact differential equation iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

and solution is given by

$$\int_{y\,constant} M dx + \int (terms\ of\ N\ independent\ of\ x) dy = constant$$

Ex : Solve $(2xy + 3y)dx + (x^2 + 3x)dy = 0$

Soln : Here $M = 2xy + 3y \text{ and } N = x^2 + 3x$

$$\frac{\partial M}{\partial y} = 2x + 3 \qquad \frac{\partial N}{\partial x} = 2x + 3$$

So that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and hence the equation is exact

∴ Solution is $\int_{y\,constant} M dx + \int (terms\ of\ N\ independent\ of\ x) dy = constant$

That is $\int (2xy + 3y)dx + \int (0)dy = c$

$\therefore 2y\frac{x^2}{2} + 3yx = c \;\; or \;\; x^2y + 3xy = c$ is the required solution

2. Solve $(2xlogy)dx + \left(\frac{x^2}{y} + 3y^2\right)dy = 0$

Soln :

Here $M = (2xlogy) \; and \; N = \left(\frac{x^2}{y} + 3y^2\right)$

$$\frac{\partial M}{\partial y} = 2x.\frac{1}{y} \quad \frac{\partial N}{\partial x} = \frac{2x}{y}$$

So that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and hence the equation is exact

$\therefore General\ solution\ is \int_{y\ const} (2xlogy)dx + \int (3y^2)dy = c$

$$\therefore x^2logy + y^3 = c$$